MATH 2050C Lecture 15 (Mar 10)

Problem Set 8 posted, due on Mar 18.

GOAL: define $\lim_{x \to c} f(x)$ where $f : A \subseteq \mathbb{R} \to \mathbb{R}$

and c is a cluster pt of A

Recall: $c \in \mathbb{R}$ is cluster pt. of $A \subseteq \mathbb{R}$

$\Leftrightarrow \forall \delta > 0, \exists x \in A$ s.t. $x \neq c$ and $|x - c| < \delta$

$\Leftrightarrow \exists$ seq. (a_n) in A s.t. $a_n \neq c \quad \forall n \in \mathbb{N}$

and $\lim(a_n) = c$

Caution: $c \in A$ OR $c \notin A$.

Def^{n}: Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function.

Suppose $c \in \mathbb{R}$ is a cluster point of A.

We say that "f converges to $L \in \mathbb{R}$ at c"

denoted: "$\lim_{x \to c} f(x) = L$" OR "$f(x) \to L$ as $x \to c$"

iff $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t.

(ie $x \neq c$)

$$|f(x) - L| < \varepsilon \quad \forall x \in A \text{ s.t. } 0 < |x - c| < \delta$$

defined

Example 1: Let $f : \mathbb{R} \to \mathbb{R}$ where (i.e. $A = \mathbb{R}$).

$$f(x) := x \qquad \forall x \in \mathbb{R}$$

Show that $\boxed{\lim_{x \to c} f(x) = c}$ $\forall c \in \mathbb{R}$.

Proof: **Observe:** Any $c \in \mathbb{R}$ is a cluster pt. of $A = \mathbb{R}$.

Fix $c \in \mathbb{R}$. Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $\delta > 0$ s.t. $\delta = \varepsilon$.

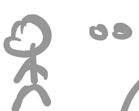
THEN, $\forall x \in A = \mathbb{R}$, $0 < |x - c| < \delta$, we have

$$|f(x) - c| = |x - c| < \delta = \varepsilon$$

$\square$

Example 2: $\boxed{\lim_{x \to c} x^2 = c^2}$ i.e. $f : A = \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

Pf: Fix $c \in \mathbb{R}$. Let $\varepsilon > 0$ be fixed but arbitrary.



Q: How to choose δ?

Suppose $0 < |x - c| < \delta$.

$$|f(x) - c^2| = |x^2 - c^2| = \underbrace{|x + c|}_{\text{bdd?}} \cdot \underbrace{|x - c|}_{\text{small}}$$

Note: $|x + c| \leq |x - c + 2c| \leq |x - c| + 2|c|$

$< \delta + 2|c|$ bdd

Suppose $|x - c| < 1$, then

$$|x+c| \le |x-c| + 2|c| < \underbrace{1 + 2|c|}_{\text{fixed!}}$$

Choose $\delta := \min\left\{ 1, \dfrac{\varepsilon}{1+2|c|} \right\}$

THEN, $\forall\, x \in A = \mathbb{R}$, $0 < |x - c| < \delta$, we have

$$|f(x) - c^2| = |x^2 - c^2| = |x+c| \cdot |x-c|$$

$$< (1+2|c|) \cdot \delta \le \varepsilon$$

$\blacksquare$

Example 3 : $\boxed{\lim_{x\to c} \dfrac{1}{x} = \dfrac{1}{c}}$ where $c \neq 0$.

Note: $f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) := \dfrac{1}{x}$

any $c \in \mathbb{R}$ is a cluster pt of A.

Pf: Fix $c \neq 0$. Let $\varepsilon > 0$ be fixed but arbitrary.

...

Suppose $0 < |x - c| < \delta \le \left\{ \dfrac{|c|}{2}, \dfrac{\varepsilon |c|^2}{2} \right\}$

$$\left| \frac{1}{x} - \frac{1}{c} \right| = \left| \frac{x-c}{xc} \right| = \frac{1}{|x|} \frac{1}{|c|} |x-c|$$

$$\le \frac{2}{|c|} \cdot \frac{1}{|c|} \cdot \delta < \varepsilon$$

<u>Note</u>: If $|x-c| < \frac{|c|}{2}$, then $|x| > \frac{|c|}{2} > 0$

<u>Picture</u>:

Choose $\delta = \min\left\{ \frac{|c|}{2}, \frac{\varepsilon |c|^2}{2} \right\} > 0$.

THEN, $\forall\ x \in A = \mathbb{R} \setminus \{0\}$, and $0 < |x-c| < \delta$,

we have

$$\left| f(x) - \frac{1}{c} \right| = \left| \frac{1}{x} - \frac{1}{c} \right| = \frac{|x-c|}{|x||c|}$$

$$= \frac{1}{|x|} \cdot \frac{1}{|c|} \cdot |x-c|$$

$$< \frac{2}{|c|} \cdot \frac{1}{|c|} \cdot \delta \leq \varepsilon$$

$\square$